International Journal of Theoretical Physics, Vol. 41, No. 9, September 2002 (© 2002)

Generalized q-Modified Laguerre Functions

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Received August 22, 2001

In this paper we define a new q-special function $A_n^\alpha(x, b, c; q)$. The new function is a generalization of the q-Laguerre function and the Stieltjes–Wigert function. We deduced all the properties of the function $A_n^\alpha(x, b, c; q)$. Finally, $\lim_{q\to 1} A_n^\alpha((1-q)x, -\beta, 1; q)$ gives $L_n^{(\alpha,\beta)}(x, q)$, which is a β-modification of the ordinary Laguerre function.

KEY WORDS: basic hypergeometric series; q-derivative operator; q-integeral; q-Laguerre function; Lie algebra.

1. INTRODUCTION

The q-shifted factorial $(a; q)_k$ (the q-extension of the Pochhammer symbol $(a)_k$) is defined by

$$(a; q)_k = \prod_{i=0}^{k-1}(1 - aq^i)$$

and satisfies the following properties:

$(qz; q)_n = \frac{(1-q^n z)}{(1-z)}(z; q)_n$
$(q^{-1}z; q)_n = \frac{(1-q^{-1}z)}{(1-q^{n-1}z)}(z; q)_n$
$(q; q)_{n-r} = (-1)^r q^{\binom{r}{2}-nr} \frac{(q;q)_n}{(q^{-n};q)_r}$

Also

$$(a; q)_\infty = \prod_{i=0}^{\infty}(1 - aq^i)$$

The basic hypergeometric series is defined by (Koekoek and Swarttouw, 1994; Koelink 1996)

$$ {}_r\varphi_s\left(\begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix} \mid q; z\right) = \sum_{k=0}^{\infty} \frac{(a_1, \ldots, a_r; q)_k}{(b_1, \ldots, b_s; q)_k} \left((-1)^k q^{\frac{k}{2}(k-1)}\right)^{1+s-r} \frac{z^k}{(q; q)_k} \quad (1)$$

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where

$$(a_1, \ldots, a_r; q)_k = \prod_{i=1}^{r}(a_i; q)_k$$

The basic hypergeometric series ${}_r\varphi_s$ is a polynomial in z if one of a_i equals q^{-n}, where n is a nonnegative integer (Chari and Pressely, 1994). Otherwise, the radius of convergence of ${}_r\varphi_s$ is

$$\rho = \begin{cases} \infty & \text{if } r < s+1 \\ 1 & \text{if } r = s+1 \\ 0 & \text{if } r > s+1 \end{cases}$$

The classical exponential function e^z can be expressed in terms of the hypergeometric functions as $e^z = {}_0F_0(\overline{}\,; z)$; this function has two different natural q-extensions denoted by $e_q(z)$ and $E_q(z)$ and defined by (Biedenharn and Lohe, 1995)

$$e_q(z) = {}_1\varphi_0\left(\begin{matrix} 0 \\ - \end{matrix} \mid q; z\right) = \sum_{k=0}^{\infty} \frac{z^k}{(q;q)_k} = \frac{1}{(z;q)_\infty} \tag{2}$$

and

$$E_q(z) = {}_0\varphi_0\left(\begin{matrix} - \\ - \end{matrix} \mid q; z\right) = \sum_{k=0}^{\infty} \frac{q^{\frac{k}{2}} z^k}{(q;q)_k} = -(z;q)_\infty \tag{3}$$

where $z \in C$, $|z| < 1$, and $0 < q < 1$. Also $e_q(z)$ and $E_q(z)$ can be considered as formal power series in the formal variable z. They have the following properties (Ahmed *et al.*, 2000):

$$e_q(z)E_q(-z) = 1$$
$$e_q(qz) = (1-z)e_q(z)$$
$$E_q(z) = (1+z)E_q(qz)$$
$$e_q(z) = (1-q^{-1}z)e_q(q^{-1}z)$$
$$E_q(q^{-1}z) = (1+q^{-1}z)E_q(z)$$
$$e_q(q^n z)E_q(-z) = (z;q)_n$$
$$e_q(z)E_q(-q^n z) = 1/(z;q)_n$$
$$\lim_{q\to1} e_q((1-q)z) = \lim_{q\to1} E_q((1-q)z) = e^z$$

2. A NEW q-SPECIAL FUNCTION $A_n^\alpha(x, b, c; q)$

Let us define the following q-special function.

Definition 1.

$$A_n^\alpha(x, b, c; q) = \frac{(bq^{\alpha+1}, q)_n}{(q,q)_n}\, {}_1\varphi_1\left(\begin{matrix} q^{-n} \\ bq^{\alpha+1} \end{matrix} \mid q; -cq^{n+b\alpha+1}x\right)$$

$$= \frac{(bq^{\alpha+1},q)_n(\frac{-c}{b}q^{(b-1)\alpha}x,q)_\infty}{(q,q)_n}\,{}_2\varphi_1\left(\begin{matrix} bq^{\alpha+n+1},0 \\ bq^{\alpha+1}\end{matrix}\mid q;\frac{-c}{b}q^{(b-1)\alpha}x\right) \tag{4}$$

Proposition 2. *The function $A_n^\alpha(x,b,c;q)$ has the generating functions*

$$(t,q)_\infty\,{}_0\varphi_2\left(\begin{matrix} - \\ bq^{\alpha+1},t\end{matrix}\mid q;-cxtq^{b\alpha+1}\right) = \sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{n}{2}}}{(bq^{\alpha+1},q)_n}t^n A_n^\alpha(x,b,c;q) \tag{5}$$

$$\frac{1}{(t,q)_\infty}\,{}_0\varphi_1\left(\begin{matrix} - \\ bq^{\alpha+1}\end{matrix}\mid q;-cxtq^{b\alpha+1}\right) = \sum_{n=0}^{\infty}\frac{A_n^\alpha(x,b,c;q)}{(bq^{\alpha+1},q)_n}t^n \tag{6}$$

$$\frac{(\gamma t,q)_\infty}{(t,q)_\infty}\,{}_1\varphi_2\left(\begin{matrix} \gamma \\ bq^{\alpha+1},t\end{matrix}\mid q;-cxtq^{b\alpha+1}\right) = \sum_{n=0}^{\infty}\frac{(\gamma,q)_n}{(bq^{\alpha+1},q)_n}A_n^\alpha(x,b,c;q)t^n \tag{7}$$

where γ is arbitrary.

Proof:

$$\begin{aligned}
{}_0\varphi_2\left(\begin{matrix} - \\ bq^{\alpha+1},t\end{matrix}\mid q;-cxtq^{b\alpha+1}\right) &= \sum_{r=0}^{\infty}\frac{(-1)^{3r}q^{3\binom{r}{2}}}{(bq^{\alpha+1},q)_r(t,q)_r}\frac{(-1)^r q^{(b\alpha+1)r}(cxt)^r}{(q,q)_r} \\
&= \sum_{r=0}^{\infty}\frac{q^{3\binom{r}{2}}q^{(b\alpha+1)r}}{(bq^{\alpha+1},q)_r(t,q)_r}\frac{(cxt)^r}{(q,q)_r}
\end{aligned}$$

But $1/(t,q)_r = e_q(t)E_q(-q^r t)$.

$$\begin{aligned}
{}_0\varphi_2\left(\begin{matrix} - \\ bq^{\alpha+1},t\end{matrix}\mid q;-cxtq^{b\alpha+1}\right) &= \sum_{r=0}^{\infty}\frac{q^{3\binom{r}{2}}q^{(b\alpha+1)r}}{(bq^{\alpha+1},q)_r}e_q(t)E_q(-q^r t)\frac{(cxt)^r}{(q,q)_r} \\
&= e_q(t)\sum_{r=0}^{\infty}\frac{q^{3\binom{r}{2}}q^{(b\alpha+1)r}}{(bq^{\alpha+1},q)_r}\sum_{n=0}^{\infty}\frac{q^{\binom{n}{2}}}{(q,q)_n} \\
&\quad\times(-q^r t)^n\frac{(cxt)^r}{(q,q)_r} \\
&= e_q(t)\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n q^{3\binom{r}{2}+(b\alpha+1+n)r+\binom{n}{2}}}{(bq^{\alpha+1},q)_r(q,q)_n} \\
&\quad\times\frac{(t)^{n+r}(cx)^r}{(q,q)_r}
\end{aligned}$$

By replacing n by $n-r$, we get

$$
{}_0\varphi_2\left(\begin{matrix} - \\ bq^{\alpha+1}, t \end{matrix} \mid q; -cxtq^{b\alpha+1}\right) = e_q(t)\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^{n-r}q^{3\binom{r}{2}+(b\alpha+1+n-r)r+\binom{n-r}{2}}}{(bq^{\alpha+1},q)_r\,(q,q)_{n-r}} \times \frac{(cx)^r(t)^n}{(q,q)_r}
$$

But $(q,q)_{n-r} = \left((q,q)_n q^{\binom{r}{2}-nr}\right)/((-1)^r(q^{-n},q)r)$.

$$
\begin{aligned}
&{}_0\varphi_2\left(\begin{matrix} - \\ bq^{\alpha+1}, t \end{matrix} \mid q; -cxtq^{b\alpha+1}\right) \\
&= e_q(t)\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n q^{2\binom{r}{2}+(b\alpha+1+2n-r)r+\binom{n-r}{2}}(q^{-n},q)_r}{(bq^{\alpha+1},q)_r\,(q,q)_n}\frac{(cx)^r(t)^n}{(q,q)_r} \\
&= e_q(t)\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n q^{2\binom{r}{2}+\frac{r}{2}(2b\alpha+3+2n-r)+\binom{n}{2}}(q^{-n},q)_r}{(bq^{\alpha+1},q)_r\,(q,q)_n}\frac{(cx)^r(t)^n}{(q,q)_r} \\
&= e_q(t)\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{r}{2}+r(b\alpha+1+n)+\binom{n}{2}}(q^{-n},q)_r}{(bq^{\alpha+1},q)_r(q,q)_n}\frac{(cx)^r(t)^n}{(q,q)_r} \\
&= e_q(t)\sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{n}{2}}t^n}{(bq^{\alpha+1},q)_n}\frac{(bq^{\alpha+1},q)_n}{(q,q)_n}\sum_{r=0}^{\infty}\frac{(-1)^r q^{\binom{r}{2}}(q^{-n},q)_r}{(bq^{\alpha+1},q)_r}\frac{(-cxq^{b\alpha+1+n})^r}{(q,q)_r} \\
&= \frac{1}{(t,q)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{n}{2}}t^n}{(bq^{\alpha+1},q)_n}A_n^{\alpha}(x,b,c;q)
\end{aligned}
$$

Similarly we can prove (2) and (3). □

Definition 3. The q-derivative operator D_q is defined by (Koornwinder, 1994)

$$
D_q f(x) = \begin{cases} \frac{f(x)-f(qx)}{x(1-q)}, & x \neq 0 \\ \frac{df(0)}{dx}, & x = 0 \end{cases}
$$

where

$$
\lim_{q\to 1} D_q f(x) = \frac{df(x)}{dx}
$$

Proposition 4. *The forward shift operator of $A_n^{\alpha}(x,b,c;q)$ is given by*

$$
A_n^{\alpha}(x,b,c;q) - A_n^{\alpha}(qx,b,c;q) = -cxq^{\alpha b+1}A_{n-1}^{\alpha+1}(q^{2-b}x,b,c;q) \tag{8}
$$

that is

$$D_q A_n^{\alpha}(x,b,c;q) = \frac{-cq^{\alpha b+1}}{1-q} A_{n-1}^{\alpha+1}(q^{2-b}x,b,c;q) \tag{9}$$

Proof:

$$\begin{aligned}
&A_n^{\alpha}(x,b,c;q) - A_n^{\alpha}(qx,b,c;q)\\
&= \frac{(bq^{\alpha+1},q)_n}{(q,q)_n}\left[{}_1\varphi_1\left(\begin{matrix} q^{-n} \\ bq^{\alpha+1}\end{matrix} \mid q; -cq^{n+b\alpha+1}x\right)\right.\\
&\quad \left. -\ {}_1\varphi_1\left(\begin{matrix} q^{-n} \\ bq^{\alpha+1}\end{matrix} \mid q; -cq^{n+b\alpha+2}x\right)\right]\\
&= \frac{(bq^{\alpha+1},q)_n}{(q,q)_n}\sum_{r=0}^{\infty}\frac{(-1)^r q^{\binom{r}{2}}(q^{-n},q)_r}{(bq^{\alpha+1},q)_r(q,q)_r}(1-q^r)(-cxq^{n+\alpha b+1})^r\\
&= \frac{(bq^{\alpha+1},q)_n}{(q,q)_n}\sum_{r=0}^{\infty}\frac{(-1)^r q^{\binom{r}{2}}(q^{-n},q)_r}{(bq^{\alpha+1},q)_r(q,q)_{r-1}}(-cxq^{n+\alpha b+1})^r\\
&= \frac{(bq^{\alpha+1},q)_n}{(q,q)_n}\sum_{r=0}^{\infty}\frac{(-1)^r q^{\binom{r}{2}}(q^{-n},q)_{r-1}(1-q^{r-n-1})}{(bq^{\alpha+1},q)_{r-1}(q,q)_{r-1}(1-bq^{\alpha+r})}(-cxq^{n+\alpha b+1})^r\\
&= \frac{(1-q^{-n})(bq^{\alpha+1},q)_n}{(1-bq^{\alpha+1})(q,q)_n}\sum_{r=0}^{\infty}\frac{(-1)^r q^{\binom{r}{2}}(q^{-n+1},q)_{r-1}}{(bq^{\alpha+2},q)_{r-1}(q,q)_{r-1}}(-cxq^{n+\alpha b+1})^r\\
&= \frac{(1-bq^{\alpha+n})}{(1-bq^{\alpha+1})}(-cxq^{\alpha b+1})\frac{(bq^{\alpha+1},q)_{n-1}}{(q,q)_{n-1}}\sum_{r=0}^{\infty}\frac{(-1)^r q^{\binom{r}{2}+r}(q^{-n+1},q)_r}{(bq^{\alpha+2},q)_r(q,q)_r}\\
&\quad \times(-cxq^{n+\alpha b+1})^r\\
&= (-cxq^{\alpha b+1})\frac{(bq^{\alpha+2},q)_{n-1}}{(q,q)_{n-1}}\,{}_1\varphi_1\left(\begin{matrix} q^{-n+1} \\ bq^{\alpha+2}\end{matrix} \mid q; -cq^{n+b\alpha+2}x\right)\\
&= -cxq^{\alpha b+1}A_{n-1}^{\alpha+1}(q^{2-b}x,b,c;q)
\end{aligned}$$

□

Proposition 5. *The backward shift operator of $A_n^{\alpha}(x,b,c;q)$ is given by*

$$\begin{aligned}
&A_n^{\alpha}(x,b,c;q) - \frac{x(1+x)q^{\alpha b}}{x-b+1}A_n^{\alpha}(qx,b,c;q)\\
&= \frac{1}{c}\left[\left(\frac{1+x}{x-b+1}+\frac{1}{xq^{\alpha b-1}}\right) A_{n+1}^{\alpha-1}(q^{b-1}x,b,c;q) - \frac{1}{xq^{\alpha b-1}}A_{n+1}^{\alpha-1}\right.\\
&\quad \left.\times(q^{b-2}x,b,c;q) - \frac{1+x}{x-b+1}\ A_{n+1}^{\alpha-1}(q^{b}x,b,c;q)\right]
\end{aligned} \tag{10}$$

that is

$$D_q\big(w(x,b,\alpha;q)A_n^{\alpha}(x,b,c;q)\big) = \frac{w(x,b,\alpha;q)}{(1-q)cx}\left[\left(\frac{1+x}{x-b+1}+\frac{1}{xq^{\alpha b-1}}\right)\right.$$
$$\times A_{n+1}^{\alpha-1}(q^{b-1}x,b,c;q)$$
$$-\frac{1}{xq^{\alpha b-1}}A_{n+1}^{\alpha-1}(q^{b-2}x,b,c;q)-\frac{1+x}{x-b+1}$$
$$\left.\times A_{n+1}^{\alpha-1}(q^{b}x,b,c;q)\right] \quad (11)$$

where

$$w(x,b,\alpha;q) = \frac{x^{\alpha b}}{(-x,(b-1)qx^{-1};q)_\infty} \quad (12)$$

Proof: At first by using

$e_q(qz) = (1-z)e_q(z)$
$e_q(q^{-1}z) = \frac{1}{(1-q^{-1}z)}e_q(z)$

we have

$$w(qx,b,\alpha;q) = \frac{(qx)^{\alpha b}}{(-qx,q^{-1}(b-1)qx^{-1};q)_\infty}$$
$$= \frac{q^{\alpha b}(1+x)}{(1-q^{-1}(b-1)qx^{-1})}\frac{x^{\alpha b}}{(-x,(b-1)qx^{-1};q)_\infty}$$
$$= \frac{q^{\alpha b}(1+x)x}{(x-b+1)}w(x,b,\alpha;q)$$

Also by using

$$A_n^{\alpha}(x,b,c;q) - A_n^{\alpha}(qx,b,c;q) = -cxq^{\alpha b+1}A_{n-1}^{\alpha+1}(q^{2-b}x,b,c;q)$$

and replacing x by xq^{b-2}, α by $\alpha-1$, and n by $n+1$, we have

$$A_n^{\alpha}(x,b,c;q) = \frac{-1}{cxq^{\alpha b-1}}\left[A_{n+1}^{\alpha-1}(q^{b-2}x,b,c;q) - A_{n+1}^{\alpha-1}(q^{b-1}x,b,c;q)\right]$$

Also replace x by qx, and then we have

$$A_n^{\alpha}(qx,b,c;q) = \frac{-1}{cxq^{\alpha b}}\left[A_{n+1}^{\alpha-1}(q^{b-1}x,b,c;q) - A_{n+1}^{\alpha-1}(q^{b}x,b,c;q)\right]$$

Now

$$D_q\big(w(x,b,\alpha;q)A_n^{\alpha}(x,b,c;q)\big)$$

$$= \frac{1}{(1-q)x}\left[w(x,b,\alpha;q)A_n^{\alpha}(x,b,c;q) - w(qx,b,\alpha;q)A_n^{\alpha}(qx,b,c;q)\right]$$

$$= \frac{w(x,b,\alpha;q)}{(1-q)x}\left[A_n^{\alpha}(x,b,c;q) - \frac{q^{\alpha b}(1+x)x}{(x-b+1)}A_n^{\alpha}(qx,b,c;q)\right]$$

$$= \frac{w(x,b,\alpha;q)}{(1-q)x}\left[-\frac{A_{n+1}^{\alpha-1}(q^{b-2}x,b,c;q) - A_{n+1}^{\alpha-1}(q^{b-1}x,b,c;q)}{cxq^{\alpha b-1}}\right.$$

$$\left. + \frac{(1+x)}{c(x-b+1)}\left\{A_{n+1}^{\alpha-1}(q^{b-1}x,b,c;q) - A_{n+1}^{\alpha-1}(q^{b}x,b,c;q)\right\}\right]$$

$$= \frac{w(x,b,\alpha;q)}{(1-q)cx}\left[\left(\frac{1+x}{x-b+1} + \frac{1}{xq^{\alpha b-1}}\right)A_{n+1}^{\alpha-1}(q^{b-1}x,b,c;q)\right.$$

$$\left. - \frac{1}{xq^{\alpha b-1}}A_{n+1}^{\alpha-1}(q^{b-2}x,b,c;q) - \frac{1+x}{x-b+1}A_{n+1}^{\alpha-1}(q^{b}x,b,c;q)\right] \qquad \square$$

Definition 6. The q-integeral on $(0,\infty)$ is defined by (Koorwinder, 1992, 1997)

$$\int_0^{\infty} f(t)\,d_qx = (1-q)\sum_{n=-\infty}^{\infty} f(q^n)q^n \tag{13}$$

Proposition 7. *The orthogonal property of the function $A_n^{\alpha}(x,b,c;q)$ is given by*

$$\int_0^{\infty} w(x,b,\alpha;q)A_n^{\alpha}(x,b,c;q)A_m^{\alpha}(x,b,c;q)\,d_qx = 0 \quad \forall n \neq m \tag{14}$$

3. SPECIAL CASES OF $A_n^{\alpha}(x,b,c;q)$

(1) At $b=0$ and $c=1$

$$A_n^{\alpha}(x,0,1;q) = \frac{1}{(q,q)_n}\,{}_1\varphi_1\left(\begin{matrix} q^{-n} \\ 0 \end{matrix} \mid q; -q^{n+1}x\right) = S_n(x,q) \tag{15}$$

where $S_n(x,q)$ is the Stieltjes–Wigert function.

(2) At $b=1$ and $c=1$

$$A_n^{\alpha}(x,1,1;q) = \frac{(q^{\alpha+1},q)_n}{(q,q)_n}\,{}_1\varphi_1\left(\begin{matrix} q^{-n} \\ q^{\alpha+1} \end{matrix} \mid q; -q^{n+\alpha+1}x\right) = L_n^{\alpha}(x,q) \tag{16}$$

where $L_n^{\alpha}(x,q)$ is the q-Laguerre function.

(3) At $x = z^2$, $b = 1$, and $c = q^{-n-\alpha}$

$$z^\alpha \lim_{q\to 1} A_n^\alpha(z^2, 1, q^{-n-\alpha}; q) = z^\alpha \frac{(q^{\alpha+1}, q)_\infty}{(q, q)_\infty} {}_1\varphi_1 \left(\begin{matrix} 0 \\ q^{\alpha+1} \end{matrix} \mid q; qz^2 \right)$$
$$= J_\alpha^{(3)}(z, q) \tag{17}$$

where $J_\alpha^{(3)}(z, q)$ is the q-Bessel function, which was introduced by Hahn and Exton and $\lim_{q\to l} J_\alpha^{(3)}((1 - q^{1/2})z, q) = J_\alpha(z)$ (Noumi and Mimachi, 1990a,b).

(4) At $b = -a^{-1}q^{-\alpha}$, $c = a^{-1}q^{-\alpha b}$, and $x \to q^{-x}$

$$A_n^\alpha(q^{-x}, -a^{-1}q^{-\alpha}, a^{-1}q^{-\alpha b}; q) = \frac{(-a^{-1}q, q)_n}{(q, q)_n} {}_1\varphi_1$$
$$\times \left(\begin{matrix} q^{-n} \\ -a^{-1}q \end{matrix} \mid q; \frac{q^{n+1-x}}{a} \right) = \frac{1}{(q, q)_n} C_n(q^{-x}, a; q) \tag{18}$$

where $C_n(q^{-x}, a; q)$ is the q-Charlier function.

(5) By using the identity

$${}_1\varphi_1 \left(\begin{matrix} q^{-n} \\ l \end{matrix} \mid q; z \right) = \frac{(q^{-1}z)^n}{(l, q)_n} {}_2\varphi_1 \left(\begin{matrix} q^{-n}; l^{-1}q^{1-n} \\ 0 \end{matrix} \mid q; \frac{q^{n+1}l}{z} \right)$$

we have

$$A_n^\alpha(x, b, c; q) = \frac{(-cxq^{\alpha b+n})^n}{(q, q)_n} {}_2\varphi_1 \left(\begin{matrix} q^{-n}; b^{-1}q^{-\alpha-n} \\ 0 \end{matrix} \mid q; -\frac{b}{cx} q^{\alpha-\alpha b+1} \right)$$

At $b = -a^{-1}q^{-\alpha-2n}$, $c = a^{-1}q^{-\alpha b-2n}$, and $x \to x^{-1}$

$$A_n^\alpha(x^{-1}, -a^{-1}q^{-\alpha-2n}, a^{-1}q^{-\alpha b-2n}; q) = \frac{(-axq^n)^{-n}}{(q, q)_n} {}_2\varphi_1$$
$$\times \left(\begin{matrix} q^{-n}; aq^n \\ 0 \end{matrix} \mid q; qx \right) = \frac{(-axq^n)^{-n}}{(q, q)_n} K_n(x, a, q) \tag{19}$$

where $K_n(x, a; q)$ is the alternative q-Charlier function.

(6) By using the identity

$${}_1\varphi_1 \left(\begin{matrix} k \\ l \end{matrix} \mid q; z \right) = \left(\frac{k}{l} z, q \right)_\infty {}_2\varphi_1 \left(\begin{matrix} k^{-1}l; 0 \\ l \end{matrix} \mid q; \frac{kz}{l} \right)$$

we have

$$A_n^{\alpha}(x, b, c; q) = \frac{(bq^{\alpha+1}, q)_n}{(q, q)_n} \left(\frac{q^{-n-\alpha-1}}{b} x, q\right)_{\infty} {}_2\varphi_1 \times \left(\begin{matrix} bq^{\alpha+n+1}; 0 \\ bq^{\alpha+1} \end{matrix} \mid q; -\frac{c}{b} x q^{\alpha b-\alpha} \right)$$

At $b = 1$, $c = -q$, $n \to -n - \alpha - 1$ and by setting $q^{\alpha} = a$

$$A^{\alpha}_{-n-\alpha-1}(x, 1, -q; q) = \frac{(aq, q)_{-n-\alpha-1}}{(q, q)_{-n-\alpha-1}} (q^n x, q)_{\infty} {}_2\varphi_1 \left(\begin{matrix} q^{-n}; 0 \\ aq \end{matrix} \mid q; qx \right)$$
$$= \frac{(aq, q)_{-n-\alpha-1}}{(q, q)_{-n-\alpha-1}} (q^n x, q)_{\infty} P_n(x, a \mid q) \tag{20}$$

where $P_n(x, a \mid q)$ is the Littel q-Laguerre/Wall function.

4. A β-MODIFICATION OF THE ORDINARY LAGUERRE POLYNOMIALS

It is well known that the four-dimensional complex Lie algebra $g(0, 1)$ with basis $J_{\pm}$, J_3, and E is defined by the commutation relations (Miller, 1968)

$$[J_{\pm}, J_3] = \pm J_{\pm} \qquad [J_+, J_-] = -E \tag{21a}$$

$$[J_{\pm}, E] = 0 \qquad [J_3, E] = 0 \tag{21b}$$

Lemma. *The Lie algebra $g(0, 1)$ has the following modified differential operators:*

$$J_+ = y\left(\frac{1}{\beta}\partial x - 1\right), \qquad J_- = \frac{1}{y}(-x\partial x - y\partial y + n) \tag{22a}$$

$$J_3 = y\partial y, \qquad E = 1 \tag{22b}$$

where $\partial x = \frac{\partial}{\partial x}$.

Proof:

$$[J_3, J_+] = J_3 J_+ - J_+ J_3$$
$$= y\partial y\left(\frac{y}{\beta}\partial x - y\right) - y\left(\frac{1}{\beta}\partial x - 1\right) y\partial y$$

$$= \frac{y}{\beta}\partial x + \frac{y^2}{\beta}\partial^2 xy - y - y^2\partial y - \frac{y^2}{\beta}\partial^2 xy + y^2\partial y$$

$$= y\left(\frac{1}{\beta}\partial x - 1\right) = J_+$$

The other commutation relations can be obtained by the same way. □

The Casimir operator of the algebra $g(0, 1)$ has the form

$$C = J_+ J_- - E J_3 \tag{23}$$

which commute with all operators. In order to obtain a realization of the above representation of $g(0, 1)$ by the operators $J_\pm$, J_3, and E, we must find nonzero functions $f_m(x, y) = y^m \phi_m(x)$, $m = 0, 1, 2, \ldots$, such that $C f_m(x, y) = 0$. From Eq. (22), we get the following differential equation:

$$x\frac{d^2\phi_m(x)}{dx^2} + (1 + m - n - \beta x)\frac{d\phi_m(x)}{dx} + \beta n\phi_m(x) = 0 \tag{24}$$

Put $m - n = \alpha$; then

$$x\frac{d^2\phi_m(x)}{dx^2} + (1 + \alpha - \beta x)\frac{d\phi_m(x)}{dx} + \beta n\phi_m(x) = 0 \tag{25}$$

This equation is a β-modification of the ordinary Laguerre equation where $\beta \in C$. The solution of the differential equation (25) is the function $L_n^{(\alpha,\beta)}(x)$, which is a β-generalization of the Laguerre function and is defined by

$$L_n^{(\alpha,\beta)}(x) = \frac{(\alpha + 1)_n}{n!}\,{}_1F_1\left(\begin{matrix} -n \\ \alpha + 1 \end{matrix} \mid \beta x\right) \tag{26}$$

We can get by some calculations the generating function

$$(1 - t)^{-\alpha-1}\ \exp\left(\frac{\beta x t}{t - 1}\right) = \sum_{n=0}^{\infty} L_n^{(\alpha,\beta)}(x) t^n \tag{27}$$

and also the orthogonality property

$$\int_0^\infty x^\alpha e^{-\beta x} L_n^{(\alpha,\beta)}(x) L_m^{(\alpha,\beta)}(x)\,dx = \frac{\Gamma(n + \alpha + 1)}{n!}\beta^{n-1}\delta_{mn} \quad \text{where } \alpha > -1 \tag{28}$$

The β-modified function $L^{\alpha,\beta}(x)$ has the following recurrence relation:

$$(n + 1)L_{n+1}^{(\alpha,\beta)}(x) - (2n - \beta x + \alpha + 1)L_n^{(\alpha,\beta)}(x) + (\alpha + n)L_{n-1}^{(\alpha,\beta)}(x) = 0 \tag{29}$$

and the normalized recurrence relation is

$$\beta x P_n(x) = P_{n+1}(x) + (2n + \alpha + 1)P_n(x) + n(\alpha + n)P_{n-1}(x) \tag{30}$$

where

$$L_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{n!} P_n(x) \tag{31}$$

The Rodrigues-type formula is given by

$$L_n^{(\alpha,\beta)}(x) = \frac{1}{n!} e^{\beta x} x^{-\alpha} \frac{d^n}{d^n x}(x^{\alpha+n} e^{-\beta x}) \tag{32}$$

Lemma. *The relation between* $A_n^{\alpha}(x, b, c; q)$ and $L_n^{(\alpha,\beta)}(x)$ *is given by*

$$\lim_{q\to 1} A_n^{\alpha}((1-q)x, -\beta, 1; q) = L_n^{(\alpha,\beta)}(x) \tag{33}$$

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